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IS $\mathcal{PT}$-SYMMETRIC QUANTUM THEORY FALSE AS A FUNDAMENTAL THEORY?

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Abstract. Yi-Chan Lee et al. claim (cf. Phys. Rev. Lett. 112, 130404 (2014)) that the “recent extension of quantum theory to non-Hermitian Hamiltonians” (which is widely known under the nickname of “$\mathcal{PT}$-symmetric quantum theory”) is “likely false as a fundamental theory”. By their opinion their results “essentially kill any hope of $\mathcal{PT}$-symmetric quantum theory as a fundamental theory of nature”. In our present text we explain that their toy-model-based considerations are misleading and that they do not imply any similar conclusions.

Keywords: quantum mechanics; PT-symmetric representations of observables, measurement outcomes; locality; quantum communication.

1. Introduction

At present it is still necessary to admit that even after almost hundred years of the study of relativistic kinematics and/or of quantum dynamics, the peaceful coexistence between our intuitive perception of the underlying classical- and quantum-physics concepts and principles is often fragile. This fragility dates back to the publication of the EPR paradox [1] and it may still be sampled by some freshmost preprints [2]. In our present paper we intend to reanalyze, critically, a re-emergence of the conflict which we noticed in one of the very recent and very well visible publications [3].

First of all, let us emphasize that the questions asked in [3] are important, with possible relevance ranging from the entirely pragmatic applications of the current quantization principles in information theory [4] up to pure mathematics [5]. In what follows we intend to complement the related discussions (to be sampled, e.g., by [6]) by a deeper analysis and re-interpretation of some technical aspects (and, mainly, the non-locality) of the toy model as used in [3].

We may briefly summarize that our analysis will support the affirmative answer to the question “Could $\mathcal{PT}$-symmetric quantum models offer a sensible description of nature?”. This conclusion will be based, first of all, on the explicit construction of all of the eligible physical inner products in all of the possible related and potentially physical, “standard” Hilbert space $\mathcal{H}^{(S)}$. In this manner, the two-parametric family of *all* of the eligible fundamental $\mathcal{PT}$-symmetric probabilistic interpretations of the system in question is constructed. In full accord with the textbooks, the observables become represented by operators which are not selfadjoint in a “false” Hilbert space but self-adjoint, as required, in another, non-equivalent, “standard” Hilbert space. Subsequently, a few implications of our construction will be discussed. In particular, it will be emphasized that the conclusions of Yi-Chan Lee et al. [3], which refer to signalling, are based on an unfortunate use of one of the simplest but still inadequate, manifestly unphysical Hilbert spaces.

2. Toy model

In letter [3] Yi-Chan Lee et al. came with a very interesting proposal of analysis of what happens, during the standard quantum entangled-state-mediated information transmission between Alice and Bob, when the Alice’s local, spatially separated part H of the total Hamiltonian (say, of $H_{tot} = H \bigotimes I$ where the identity operator I represents the “Bob’s”, spatially separated component) is chosen in the well known $\mathcal{PT}-$symmetric two-level toy-model form

$$H = s \begin{pmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha \end{pmatrix}, \quad s, \alpha \in \mathbb{R}. \tag{1}$$

The conclusions of [3] look impressive (see, i.a., the title “Local $\mathcal{PT}$ symmetry violates the no-signaling principle”). Unfortunately, many of them (like, e.g., the very last statement that the “results essentially kill any hope of $\mathcal{PT}$-symmetric quantum theory as a fundamental theory of nature”) are based on several unfortunate misunderstandings. In what follows we intend to separate the innovative and inspiring aspects of the idea from some of the conclusions of [3] which ignore the overall non-locality of the toy model and which must be classified as strongly misleading and/or inadequate if not plainly incorrect.

2.1. $\mathcal{PT}-$symmetry

Our task will be simplified by the elementary nature of the toy-model Hamiltonian H of (1) with property $H\mathcal{PT} = \mathcal{PT}H$ called $\mathcal{PT}-$symmetry (for the sake of clarity let us recall that one may choose here operator $\mathcal{P}$ in the form of Pauli σ_x matrix while $\mathcal{T}$ may be defined simply as complex conjugation). Secondly, our task will be also simplified by the availability of several published reviews of the formalism (let us call it $\mathcal{PT}-$symmetric Quantum Mechanics, PTQM) and, in particular, of its recent history of development (let us recall its most exhaustive descriptions [5, 7, 8]).

Incidentally, it is extremely unfortunate that the latter three PTQM summaries remained, obviously, unknown to or, at least, uncited by, the authors of letter [3] (for the sake of brevity let us call this letter "paper I" in what follows). Otherwise, the authors of paper I would be able to replace their first and already manifestly incorrect description of the birth of the formalism (in fact, the first sentence of their abstract which states that in 1998, "Bender et al. [9] have developed $\mathcal{PT}$-symmetric quantum theory as an extension of quantum theory to non-Hermitian Hamiltonians") by some more appropriate outline of the history. Reminding the readers, e.g., that for the majority of active researchers in the field (who are meeting, every year, during a dedicated international conference [10]) the presently accepted form of the $\mathcal{PT}$-symmetric quantum theory has only been finalized, roughly speaking, after the publication of the "last erratum" [11] in 2004.

Naturally, even the year 2004 was not the end of the history since during 2007, for example, the description of the so called $\mathcal{PT}$-symmetric brachistochrone [12] moved a bit out of the field and had to be followed by the thorough (basically, open-system-related) re-clarification of the concept (cf. [13] and also, a year later, [14, 15]). During the same years also the methods of extension of the intrinsically non-local PTQM formalism to the area of scattering experiments were developed [16–18].

2.2. Eligible physical inner products

Unfortunately, the authors of paper I have missed the latter messages. Having restricted their attention solely to the brachistochronic quantum-evolution context of the initial publication [12] they remained unwarned that in this context the role of the generator H may be twofold. It is being used *either* in the *unitary* quantum-evolution context of [7, 8] (cf. also the highly relevant, cca fifteen years older review paper [19]) *or* in application to the open quantum systems.

In the latter case one is allowed to speak just about a *non-unitary*, truly brachistochronic quantum evolution *within a subspace* of a "full" Hilbert space of states [14, 15]. Naturally, the quantum world of the above-mentioned Alice cannot belong to such a category. In other words, "her" Hamiltonian (1) *must necessarily* be made self-adjoint. According to the standard theory (briefly reviewed also in our compact review [20]), this should be made via a replacement of the "friendly but false" Hilbert space $\mathcal{H}^{(F)}$ (chosen, in paper I, as $\mathcal{H}^{(F)} \equiv \mathbb{C}^2$ for model (1)) by another, "standard, sophisticated" Hilbert space $\mathcal{H}^{(S)}$ which only differs from $\mathcal{H}^{(F)}$ in its use of a *different, locality-violating inner product* between its complex two-dimensional column-vector elements $|a\rangle = (a_1, a_2)^T$ and $|b\rangle = (b_1, b_2)^T$.

The usual and "friendly", $F-$superscripted inner product

$$\langle a|b\rangle = \langle a|b\rangle^{(F)} = \sum_{i=1,2} a_i^* b_i \tag{2}$$

defines the Hilbert-space structure in the false and manifestly unphysical, ill-chosen and purely auxiliary friendlier space $\mathcal{H}^{(F)}$. Thus, what is now required by the PTQM postulates is an introduction of a *different, non-local*, $S-$superscripted product

$$\langle a|b\rangle^{(S)} = \sum_{i,j=1,2} a_i^* \Theta_{ij} b_j \tag{3}$$

containing an *ad hoc* (i.e., positive and Hermitian [19]) "Hilbert-space-metric" matrix $\Theta = \Theta^{(S)}$. Precisely this enables us to reinterpret our given Hamiltonian H with real spectrum as living in a manifestly physical, new Hilbert space $\mathcal{H}^{(S)}$. Naturally, one requires that such a Hamiltonian generates a unitary evolution in the correct, physical Hilbert space $\mathcal{H}^{(S)}$ or, in mathematical language, that it becomes self-adjoint with respect to the upgraded inner product (3).

3. Physics

3.1. Admissible probabilistic interpretations of the model

For our two-dimensional matrix model (1) the latter condition proves equivalent to the set

$$H^\dagger \Theta = \Theta H \tag{4}$$

of four linear algebraic equations with general solution

$$\Theta = a^2 \begin{pmatrix} 1 & u - i\sin\alpha \\ u + i\sin\alpha & 1 \end{pmatrix}, \qquad a, u, \alpha \in \mathbb{R}, \quad |u| < |\cos\alpha|. \tag{5}$$

Any choice of admissible parameter u is easily shown to keep this metric (as well as its inverse) positive. Thus, the reason why the parameter α was called "the non-Hermiticity of H" in [14] is purely conventional, based on a tacit assumption that one speaks, say, about an open quantum system. On the contrary, once we restrict our attention to the world of Alice (who *must* live in the *physical* Hilbert space $\mathcal{H}^{(S)}$), we *must* speak about the *unitarily* evolving quantum states and about the relevant generator (1) which is, by construction, *Hermitian* inside any pre-selected physical Hilbert space, given by our choice of the free parameter u.

For this reason the calculation of what, according to paper I, "Bob will measure using conventional quantum mechanics" must be again performed in the physical Hilbert space. In particular, the trace formulae as used in paper I are incorrect and must be complemented by the pre-multiplication of the bra vectors by the "shared" metric from the right, $\langle\psi_f| \to \langle\psi_f|\tilde{\Theta}$ (in the most elementary scenario one could simply recall (5) and choose $\tilde{\Theta} = \Theta \bigotimes I$).

3.2. OBSERVABLES

Many of the related comments in paper I (like, e.g., the statement that "These states [given in the unnumbered equation after (2)] are not orthogonal to each other in conventional quantum theory") must be also modified accordingly. The point is that the non-orthogonality of the eigenstates of H in the manifestly unphysical Hilbert space $\mathcal{H}^{(F)}$ is entirely irrelevant. In contrast, what remains decisive and relevant is that, in the words of paper I, "when $\alpha = \pm\pi/2$, they become the same state, and this is the $\mathcal{PT}$ symmetry-breaking point". Indeed, one easily checks that in such an "out-of-theory" limit (towards the so called Kato's exceptional point [21]) the metric (and, hence, the physical Hilbert space) ceases to exist.

One has to admit that the currently accepted PHQP terminology is a bit unfriendly towards newcomers. Strictly, one would have to speak about the Hermiticity of any two-by-two matrix observable Λ, i.e., equivalently, about the validity of the necessary Hermiticity condition in physical space,

$$\Lambda^\dagger\Theta = \Theta\Lambda. \tag{6}$$

Naturally, this condition can only be tested *after* we choose a definite form of the metric (5), i.e., in our toy model, after we choose the inessential scale factor $a^2 > 0$ and the essential metric-determining parameter u in (5).

It is worth adding that in order to minimize possible confusion the authors of the oldest review paper [19] recommended that, firstly, whenever one decides to work with a nontrivial (sometimes also called "non-Dirac") metric $\Theta \neq I$, the natural Hermiticity condition in the "hidden" physical space should be better called "quasi-Hermiticity". Secondly, they also recommended that having a Hamiltonian, there may still be reasons for our picking up a suitable candidate Λ for another observable in advance. Then, equation (6) would acquire a new role of an additional phenomenological constraint imposed upon the metric.

Incidentally, in the PTQM context the latter idea found its extremely successful implementation in which one requires that the second observable Λ represents a charge of the quantum system in question. It is rather amusing to verify that such a specific requirement (called, sometimes, $\mathcal{PCT}$ symmetry [7]) would remove all of the ambiguities from the metric of (5) simply by fixing the value of $u = 0$ as well as of $a^2 = 1/\cos\alpha$.

4. CONCLUSIONS

We are now prepared to return to the two key PTQM assumptions as formulated in paper I. Their main weakness is that they use the concept of the physical Hilbert space (i.e., in essence, of the unitarity of evolution) in a very vague manner. One should keep in mind that even in the phenomenologically extremely poor two-dimensional toy models the predictions and physical content of the theory are very well understood as given not only by the generator of evolution H but also by the second observable Λ (say, charge — for both, naturally, we require that the spectrum is real). Thus, what can be measured in the model is the energy and, say, charge. In other words, the theory does not leave any space for any kind of coexistence between different "conventional" metrics and/or between different normalization conventions (i.e., typically, for the simultaneous use of different parameters u in (5)). At the same time, in the light of paper [18] on the PTQM-compatible unitarity of the scattering, the PTQM theory still leaves space for a consistent implementation of the important phenomenological concepts like locality, etc.

The concluding remarks of paper I about a conjectured "trichotomy of possible situations" must be thoroughly reconsidered. Keeping in mind the necessary separation of alternative PTQM-related problems and eliminating, first of all, any mixing between the two well defined categories, viz., of the quantum models characterized by the unitary and/or non-unitary evolution. Definitely, the theories of the PTQM type did not exhaust their potentialities yet. It is truly impossible to agree with the final statement of paper I that its "results essentially kill any hope of $\mathcal{PT}$-symmetric quantum theory as a fundamental theory of nature".

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